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# 行列群の連続有限型因子環上への非コサイクル同値な連続個の作用

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## 1. 前置き

ここでは、行列のなす群に対して、有限連続型因子環  $M$  の自己同型写像群としての、一連の表現  $\alpha_s, (s \in [0, 1/2])$  を、与える。

ここでの因子環  $M$  としては、次のようなものが現れる：

- i) 近似的有限次元型環  $R$
- ii) 自由群  $F_n, (n = 2, 3, \dots, \infty)$  の左正則表現により生成されるノイマン環  $L(F_n)$ .

i) の場合には、行列群  $G$  として  $SL(n, \mathbb{Z}), n \geq 3, Sp(n, \mathbb{Z}), n \geq 2$  を取れば、接合積  $R \times_{\alpha_s} G$  は Kazhdan の性質 T を持つ因子環となる。ところが、ii) の場合、特に  $n = \infty$  の時には、同じ群をもってきても、この接合積は、因子環となるが、Kazhdan の性質 T を持つことは、出来ない。

全ての場合において、これらの連続個の表現は 互いに異なる  $s$  と  $t$  に対して  $\alpha_s$  と  $\alpha_t$  とは互いに非共役である。又、とくに、行列群が特殊群  $SL(n, \mathbb{Z}), n \geq 3$  のときには、更にコサイクル同値に、なることもできない連続個の表現である。

更に、これらの表現に現れる個々の自己同型写像  $\alpha_s(T)$  に対する Connes-Stormer のエントロピー  $H(\alpha_T^s)$  は次のような関係式をみたす。

$$\log(\max\{\prod_{i=1}^n \mu_i, \prod_{i=1}^n \mu'_i\}) \leq H(\alpha_T^s) \leq \log(\prod_{i=1}^n \mu_i \mu'_i)$$

となる。ただし、 $T$  は可逆な  $n$  次正方行列で、その固有値  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  に対して、

$$\mu_i = \max(|\lambda_i|, 1), \quad \mu'_i = \max\left(\frac{1}{|\lambda_i|}, 1\right)$$

である。

## II. 行列群の連続有限型因子環上への作用

この章では、行列群  $GL(m, \mathbb{Z})$ ,  $m \geq 2$  の超有限連続有限型因子環  $R$  と  $n$  ( $2 \leq n \leq \infty$ ), 個の生成元をもつ自由群  $F_n$  の左正則表現によって与えられる因子環  $L(F_n)$  の自己同型写像群としての作用を、与える。

### 2.1. 自由積に対するシンプレティック 形式

先ず通常のシンプレティック 形式 の概念を拡張して、自由積に対するシンプレティック 形式を定義する。

**2.1.1.** Let  $n$  be a positive integer. Let  $B(a, b)$  be the symplectic form of  $a$  and  $b$  in the vector space  $\mathbb{R}^{2n}$  ( $\mathbb{R}$  is the set of all real numbers) :

$$B(a, b) = \sum_{i=1}^n a_i b_{n+i} - \sum_{i=1}^n a_{n+i} b_i,$$

where  $a = (a_1, \dots, a_n, a_{n+1}, \dots, a_{2n}) \in \mathbb{R}^{2n}$ . Let  $E_n$  be the identity matrix of the order  $n$  and let  $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ . Then  $B(a, b) = (a, Jb)$ , the natural inner product of  $\mathbb{R}^{2n}$ . The symplectic group  $Sp(n, \mathbb{R})$  is the set of a matrix  $T$  such that  $B(a, b) = B(Ta, Tb)$  for all  $a, b \in \mathbb{R}^{2n}$ .

**2.1.2.** We denote by  $A_i$  a copy of the additive group  $\mathbb{Z}^{2n}$  for all  $i \in \mathbb{Z}$ . Let  $I \subset \mathbb{Z}$  be a subset, and let  $A(I) = \ast_{i \in I} A_i$  be the free product group. Each  $a \in A(I)$  is expressed by the unique form that

$$(*) \quad a = a_1 a_2 \cdots a_p, \quad a_i \in A_{\iota_i}, a_i \neq 0 \quad \iota_i \in I, \quad \iota_1 \neq \iota_2 \neq \cdots \neq \iota_p.$$

The  $p$  is called the length of  $a$  and the set  $\{\iota_1, \iota_2, \dots, \iota_p\}$  is called the alphabet for the word  $a$ , which we denote by  $J(a)$ . Given  $i \in J(a)$ , let  $I(a, i) = \{j \in I : \iota_j = i\}$ .

Let  $\Phi_i : A(I) \rightarrow A_i$  be the homomorphism defined by

$$\Phi_i(a) = \sum_{j \in I(a, i)} a_j,$$

for  $a \in A(I)$  with the reduced form (\*). If  $I(a, i) = \emptyset$ , we let  $\Phi_i(a) = 0$ .

Define a homomorphism  $\alpha : sp(n, \mathbb{Z}) \rightarrow \text{Aut}(A(I))$  by

$$\alpha_T(a) = T(a_1)T(a_2) \cdots T(a_p), \quad \text{for } T \in Sp(n, \mathbb{Z}).$$

Here  $Sp(n, \mathbb{Z})$  is the symplectic group whose components are integers, and  $a \in A(I)$  has the form (\*).

**2.1.3.** Let

$$B_I(a, b) = \sum_{i \in J(a) \cap J(b)} B(\Phi_i(a), \Phi_i(b)), \quad a, b \in A(I).$$

In the case where  $J(a) \cap J(b) = \emptyset$ , we let  $B_I(a, b) = 0$ . Then we have the following proposition by calculations.

**2.1.4. Proposition.** *The following basic properties of the symplectic form  $B(\cdot, \cdot)$  in 2.1.1 are satisfied ;*

$$\left\{ \begin{array}{l} B_I(a, 1) = B_I(1, a) = B_I(a, a^{-1}) = 0, \\ B_I(a, b) = -B_I(b, a), \\ B_I(ab, c) = B_I(a, c) + B_I(b, c), \\ B_I(a, b) = B_I(\alpha_T(a), \alpha_T(b)), \end{array} \right.$$

where  $a, b, c \in A(I)$ ,  $T \in sp(n, \mathbb{Z})$  and 1 is the identity of the group  $A(I)$ , and

$$B_I(a, b) = 0, \quad \text{if } a \in A_i, b \in A_j, i \neq j.$$

## 2.2. 自由積群に対する 2-コサイクル.

**2.2.1.** Let  $A(I) = \ast_{i \in I} A_i$  be the same group as in the section 2.1. We consider the semidirect product

$$G(I) = sp(n, \mathbb{Z}) \times_s A(I),$$

where the product of whose elements are defined by

$$(S, a)(T, b) = (ST, \alpha_T^{-1}(a)b), \quad (S, T \in sp(n, \mathbb{Z}), a, b \in A(I)).$$

Let  $s \in [0, \pi/2]$  be irrational (*mod*.  $2\pi$ ). Put

$$\mu_s((S, a), (T, b)) = e^{\sqrt{-1} \, s\pi B_I(\alpha_T^{-1}(a), b)}, \quad (S, T \in sp(n, \mathbb{Z}), a, b \in A(I)).$$

Then we have the following Proposition using the symplectic property of  $B_I(\cdot, \cdot)$ .

**2.2.2. Proposition.** *The  $\mu_s$  is a normalized 2-cocycle of  $G(I) \times G(I)$  to the torus*

$\mathbb{T}$  :

$$\begin{cases} \mu_s(1, g) = \mu_s(g, 1) = \mu_s(g, g^{-1}) = 1 \\ \mu_s(f, g)\mu_s(fg, h) = \mu_s(g, h)\mu_s(f, gh) \end{cases}$$

## 2.3. 自由群因子環 $L(F_n)$ への $GL(n, \mathbb{Z})$ の作用.

上で、準備したことを用いて、自由群因子環  $L(F_n)$  への  $GL(n, \mathbb{Z})$  の作用を、与える。この与え方は、下の命題 2.3.2 で、示すように、特殊な場合には、超有限連続有限型因子環  $R$  への作用を、同時に与えることになることを、注意する。

**2.3.1.** Let  $\mu_s$  be the normalized 2-cocycle defined in 2.2. The left  $\mu_s$ -representation  $\lambda^s$  of the group  $G(I) = sp(n, \mathbb{Z}) \times_s A(I)$  is defined by

$$(\lambda^s(g)\xi)(h) = \mu_s(h^{-1}, g)\xi(g^{-1}h), \quad g, h \in G(I), \xi \in l^2(G(I)).$$

Then  $\lambda^s$  is a  $\mu_s$ -cocycle unitary representation of  $G(I)$  on  $l^2(G(I))$  :

$$\lambda^s(g)\lambda^s(h) = \mu_s(g, h)\lambda^s(gh).$$

We denote by  $N_s(I)$  the von Neumann algebra generated by  $\lambda^s(A(I))$ .

Let  $L(F_k)$  be the von Neumann algebra generated by the left regular representation of the free group  $F_k$  with  $k$  generators,  $k \geq 2$ .

**2.3.2. Proposition.** *Let  $|I|$  be the cardinality of the set  $I$ .*

- (1) *If  $|I| = 1$ , then  $N_s(I)$  is isomorphic to the hyperfinite  $II_1$  factor  $R$ .*
- (2) *If  $|I| \geq 2$ , then the von Neumann algebra  $N_s(I)$  is isomorphic to the free group factor  $L(F_{|I|})$ .*

Define the homomorphism  $\alpha^s : sp(n, \mathbb{Z}) \rightarrow \text{Aut}(N_s(I))$  by

$$\alpha_T^s(x) = \lambda^s(T, 1)x\lambda^s(T, 1)^*, \quad (T \in sp(n, \mathbb{Z}), x \in N_s(I)).$$

Then we have

$$\alpha_T^s(\lambda^s(1, a)) = \lambda^s(1, \alpha_T(a)), \quad (T \in sp(n, \mathbb{Z}), a \in A(I)).$$

Let  $\eta$  be the imbedding of  $GL(n, \mathbb{R})$  into  $sp(n, \mathbb{R})$  given by

$$\eta(T) = \begin{pmatrix} T & 0 \\ 0 & (T^t)^{-1} \end{pmatrix}.$$

Here  $T^t$  means the transposed matrix of  $T$ .

**2.3.3. Definition.** We define the action  $\alpha^s$  of  $GL(n, \mathbb{Z})$  on  $N_s(I)$  by the homomorphism  $\alpha^s \circ \eta : GL(n, \mathbb{Z}) \rightarrow \text{Aut}(N_s(I))$ .

Let  $M$  be a  $II_1$  factor with the unique trace  $\tau$ . and let  $G$  be a discrete group. An action  $\beta$  of  $G$  on  $M$  is "outer" if  $\beta_g$  is an outer automorphism for all  $g \in G$ , and  $\beta$  is "mixing" if given  $x, y \in M$  and  $\epsilon > 0$  there exists a finite subset  $K$  of  $G$  such that

$$|\tau(y\beta_g(x)) - \tau(x)\tau(y)| < \epsilon, \quad \forall g \notin K.$$

Remark that if  $\beta$  is mixing, then  $\beta$  is ergodic.

**2.3.4. Proposition.** Let  $G \subset GL(n, \mathbb{Z})$ ,  $(n \geq 2)$  be a non-trivial subgroup.

(1) The action  $\alpha^s : G \rightarrow \text{Aut}(N_s(I))$  is outer.

(2) If given finite subsets  $S_1, S_2 \subset \mathbb{Z}^{2n}$ ,  $G$  contains a finite subset  $K$  such that

$$S_1 \cap \{\eta(T)a; a \in S_2\} = \emptyset, \quad \forall T \notin K,$$

then  $\alpha^s$  is mixing for all  $s$ .

Let  $\{e_j; 1 \leq j \leq 2n\}$  be the standard basis in  $\mathbb{Z}^{2n}$ , whose copy in  $A_i$  we denote by  $\{e(i; j); 1 \leq j \leq 2n\}$ . Let  $S$  be a subset of  $\{1, 2, \dots, n\}$  and let

$$A_S = \{a \in A(I); a = a_1 a_2 \cdots a_p, a_k \in \cup_{l \in S} \mathbb{Z} e(\iota_k, l), \iota_1 \neq \cdots \neq \iota_p\}.$$

**2.3.5. Lemma.** Let  $G \subset GL(n, \mathbb{Z})$  be a subgroup, and let  $\theta$  be an isomorphism of  $N_s(I)$  onto  $N_t(I)$  such that  $\theta \alpha_T^s = \alpha_T^t \theta$  for all  $T \in G$ .

(1) Let us fix an integer  $j$  with  $1 \leq j \leq n$ .

(1-1) Assume that  $n \geq 3$ : If  $G$  contains matrices  $\{T_{1,l}, T_{2,k}; 1 \leq l, k \leq n, l \neq j, k \neq n\}$  whose  $p, q$  component  $T(p, q)$  satisfies that  $T_{1,l}(p, p) = 1$  for all  $p$ ,  $T_{1,l}(j, l) \neq 0$  and  $T_{1,l}(p, q) = 0$  otherwise, and  $T_{2,k}(p, p) = 1$  for all  $p$ ,  $T_{2,k}(k, n) \neq 0$  and  $T_{2,k}(p, q) = 0$  otherwise, then for all  $i \in I$  we have the Fourier expansion :

$$\theta(\lambda^s(e(i; j))) = \sum_{a \in A_j} c(a) \lambda^t(a), \quad (c(a) \in \mathbb{C}), \text{ in the } \|\cdot\|_2 \text{ convergence topology.}$$

Moreover if  $G$  contains a matrix  $T_3$  such that  $T_3(j, j) = 1, T_3(p, j) \neq 0$  for some  $p \neq j$ , then there exist a permutation  $\sigma$  of  $I$  and an  $m_j \in \mathbb{Z}$  so that

$$\theta(\lambda^s(e(i; j))) = \lambda^t(m_j e(\sigma(i), j)), \quad \forall i \in I.$$

(1-2) Case that  $n = 2$ : If  $G$  contains the above matrix  $\{T_{1,l}; l \neq j\}$ , then for all  $i \in I$  we have the Fourier expansion for  $S = \{j, 5-j\}$  :

$$\theta(\lambda^s(e(i; j))) = \sum_{a \in A_S} c(a) \lambda^t(a), \quad (c(a) \in \mathbb{C}), \text{ in the } \|\cdot\|_2 \text{ convergence topology.}$$

Moreover if  $G$  contains the above matrix  $T_3$ , then there exist a permutation  $\sigma$  of  $I$ ,  $c \in \mathbb{T}$  and an  $m_k \in \mathbb{Z}$ , ( $k = 1, 2$ ) so that

$$\theta(\lambda^s(e(i; j))) = c_j \lambda^t(m_1 e(\sigma(i), j)) \lambda^t(m_2 e(\sigma(i), 5 - j)).$$

(2) Let us fix an integer  $j$ , ( $n + 1 \leq j \leq 2n$ ). Same formulas hold if conditions are satisfied by replacing the matrices  $T$  to  $(T^t)^{-1}$ .

**2.3.6. Proposition.** Let  $G \subset GL(n, \mathbb{Z})$ , ( $n \geq 2$ ) be a subgroup which contains the matrices in Lemma 2.3.4 for all  $j$ . Then  $\alpha^s : G \rightarrow \text{Aut}(N_s(I))$  is not conjugate to  $\alpha^t : G \rightarrow \text{Aut}(N_t(I))$  if  $s \neq t$ .

**2.3.7. Corollary.** The groups  $SL(n, \mathbb{Z})$ ,  $GL(n, \mathbb{Z})$ , ( $n \geq 2$ ) and the free group  $F_2$  have a continuous family of non-conjugate mixing outer actions on the free group factor  $L(F_m)$  for all  $m = 2, 3, \dots, \infty$ .

## 2.4. コホモロジー類 とカズダンの性質 T

この節では、特に、行列群のうちでも、カズダンの性質 T を持つ群を中心に取り扱う。

Let  $\alpha$  be an ergodic action of a discrete group  $G$  on a  $\text{II}_1$  factor  $M$ , and let  $U(M)$  be the unitary operators of  $M$ . We denote by  $Z_{\alpha, \text{erg}}^1$  the set of 1-cocycle unitary representation  $u$  of  $G$  on  $M$  for  $\alpha$  such that  $\text{Ad}(u_g) \cdot \alpha_g$  is also ergodic :

$$Z_{\alpha, \text{erg}}^1 = \{u : G \rightarrow U(M) \mid \text{Ad}(u_g) \circ \alpha_g \text{ is ergodic, } u_g \alpha_g(u_h) = u_{gh}, \forall g, h \in G, \}.$$

Two cocycles  $u_g, v_g$  are said to be cohomologous and denoted by  $u_g \sim v_g$  if there exists a unitary  $u \in M$  such that  $u_g = uv_g \alpha_g(u^*)$ . Let

$$H_{\alpha, \text{erg}}^1 = Z_{\alpha, \text{erg}}^1 / \sim.$$



**2.4.1. Theorem.** *Let  $G \subset GL(n, \mathbb{Z})$ ,  $(n \geq 2)$  be a subgroup which contains the matrices in Lemma 2.3.5 for all  $j$ ,  $(1 \leq j \leq n)$ . If  $G$  has the property  $T$  of Kazhdan, then the actions  $\alpha^s : G \rightarrow \text{Aut}(L(F_m))$  gives a continuous family, any two of which are not cocycle conjugate for all  $m = 2, 3, \dots, \infty$ .*

**2.4.2. Corollary.** *Each of the group  $SL(n, \mathbb{Z})$ ,  $n \geq 3$  and  $Sp(n, \mathbb{Z})$ ,  $n \geq 2$  has a continuous family of ergodic outer actions on the free group factor  $L(F_m)$ ,  $m \geq 2$  such that each two of them are not cocycle conjugate.*

*Proof.* The group  $SL(n, \mathbb{Z})$ ,  $n \geq 3$  and  $Sp(n, \mathbb{Z})$ ,  $n \geq 2$  have the property  $T$  of Kazhdan by [K, DK], and satisfy the conditions in Theorem 2.4.1.  $\square$

## 2.5. 上記の作用 $\alpha^s$ に関する接合積環の性質

Let us consider the crossed product  $M_s(n, I) = N_s(I) \rtimes_{\alpha^s} SL(n, \mathbb{Z})$ . Then the von Neumann algebra  $M_s(n, I)$  is generated by  $\lambda_s(\eta(SL(n, \mathbb{Z})) \times_s A(I))$ . Moreover,  $M_s(n, I) \cong R \rtimes_{\alpha^s} SL(n, \mathbb{Z})$  if  $|I| = 1$ , where  $R$  is the hyperfinite  $\text{II}_1$  factor, and  $M_s(n, I) \cong L(F_m) \rtimes_{\alpha^s} SL(n, \mathbb{Z})$  if  $2 \leq |I| = m \leq \infty$ . Since the action  $\alpha^s$  is outer by Proposition 2.3.4, the  $M_s(n, I)$  is a type  $\text{II}_1$  factor.

**2.5.1.** In this section, we remark that the crossed products  $M_s(2, I) = N_s(I) \rtimes_{\alpha^s} SL(2, \mathbb{Z})$  is a factor which have "HT free group subfactor  $L(F_m)$ " if  $|I| = m \geq 2$ . The notion of "HT free group subfactor  $L(F_n)$ " is a modification of HT Cartan subalgebra in the sense of Popa ([Po2]) as follows :

Let  $M$  be a finite von Neumann algebra, and let  $B \subset M$  be a von Neumann subalgebra. The embedding  $B \subset M$  has *the Property T*, if it has the property which is a notion in von Neumann algebra text of Margulis' property  $T$  ([M], cf. [dHV]) for the pair of groups, that is, there exists a finite subset  $\{x_1, x_2, \dots, x_n\}$

of  $M$  and  $\epsilon > 0$  such that if  $H$  is a Hilbert  $M$  bimodule with  $\xi \in H$  a unit vector which satisfies that  $\|x_i \xi - \xi x_i\| \leq \epsilon$  for all  $i$ , then there exists a non zero vector  $\xi_0 \in H$  such that  $b\xi_0 = \xi_0 b$  for all  $b \in B$ . When  $M$  is a type  $\text{II}_1$  factor, the embedding  $B \subset M$  is said to have *the property H* if it has a property which is a generalization of Hagerup's compact approximation property ([Po2 : Definition 2.3]), and  $B \subset M = B \rtimes_\sigma G$  has the property H if  $G$  has positive definite functions  $\phi_n$  such that

$$\phi_n(1) = 1, \quad \lim_{g \rightarrow \infty} \phi_n(g) = 0, (\forall n), \quad \lim_{n \rightarrow \infty} \phi_n(g) = 1, (\forall g \in G).$$

Furthermore, an abelian  $C^*$ -subalgebra  $B$  of a type  $\text{II}_1$  factor  $M$  is called a *HT Cartan subalgebra* of  $M$  if it satisfies the following conditions :

- 1)  $B' \cap M = B$  and  $N_M(B) = \{\text{unitary } u \in M : uBu^* = B\}$  generates  $M$ .
- 2)  $B \subset M$  has the property H.
- 3)  $B$  has a von Neumann subalgebra  $B_0 \subset B$  such that  $B'_0 \cap M = B$  and such that  $B_0 \subset M$  has the property T.

Popa remarked about "HT hyperfinite subfactor"  $R$  of the factor  $R \rtimes_\sigma G_0$  in [Po2 : Remark 6.6].

Here, we consider a notion corresponding HT Cartan subalgebra for subfactors which is isomorphic to the free group factor  $L(F_n), n \geq 2$ . We say that a subfactor  $Q$  of a type  $\text{II}_1$  factor  $M$  is a *HT free group subfactor* of  $M$  if it satisfies the following conditions :

- 1)  $Q' \cap M = \mathbb{C}$  and  $N_M(Q)$  generates  $M$ .
- 2)  $Q \subset M$  has the property H.
- 3)  $Q$  has a von Neumann subalgebra  $Q_0 \subset Q$  such that  $Q'_0 \cap M = Q'_0 \cap Q$  and such that  $Q_0 \subset M$  has the property T.

**2.5.2. Proposition.** Assume that  $|I| \geq 2$ . The type  $II_1$  factor  $M_s(2, I)$  has a HT free group subfactor isomorphic to  $L(F_{|I|})$  for all  $s \in [0, \pi/2]$  mod.  $2\pi$ .

Remark by the same proof that  $R = N_s(I)$  is a HT hyperfinite subfactor of  $M$  when  $|I| = 1$ .

**2.5.3. Remark.** Assume that  $n \geq 3$ . As we showed in [Ch2], if  $|I| = 1$ , then the factor  $M_s(n, I)$  has property T of Connes-Jones ([CJ]) because the group  $\eta(SL(n, \mathbb{Z})) \times_s \mathbb{Z}^{2n}$  has property T of Kadhdan. If  $|I| = \infty$ , then the group  $\eta(SL(n, \mathbb{Z})) \times_s A(I)$  does not have property T because it has infinite generators and can not have property T by [K : Theorem 2], so that the factor  $M_s(n, I)$  does not have property T by [CJ : Theorem 2]. We don't know whether  $M_s(n, I)$  has property T or not, in the case where  $1 \neq |I| < \infty$ .

### III. 作用の中に現れる個々の自己同型写像のエントロピーの値

The each automorphism of  $L(F_m)$ ,  $m \geq 2$  in the actions that we discussed in the section 2 is given essentially as the free products of those in [Ch2]. Using this fact, in this section, we give an estimation of the Connes-Størmer entropy  $H(\alpha_T^s)$  for each automorphism  $\alpha_T^s$ , ( $T \in GL(n, \mathbb{Z})$ ) of the type  $II_1$  factor  $N_s(I)$ . The cocycle actions of Popa are given as the reduced actions of the free permutation  $\sigma \in \text{Aut}(L_\infty)$ , and it is known that  $H(\sigma) = 0$  ([S1], cf. [S2, BC, D4]).

#### 3.1. Entropy $ht_\phi(\alpha)$

To obtain an estimation of the values for the entropies, we need an entropy defined in [Ch5], which is a slight modification of Voiculescu's topological entropy ([V2], cf. [Br]). First, we review the definition and basic properties of the entropy

By a  $C^*$ -dynamical system  $(A, \alpha, \phi)$ , we mean that  $A$  is a separable unital  $C^*$ -algebra,  $\alpha$  is a  $*$ -automorphism of  $A$  and  $\phi$  is an  $\alpha$ -invariant state of  $A$ .

**3.1.1.** Given a  $C^*$ -dynamical system  $(A, \alpha, \phi)$ , let  $\pi$  be a faithful  $*$ -representation of  $A$  on a Hilbert space  $H$ , and let  $\xi \in H$  be a unit vector such that  $\phi = \omega_\xi \circ \pi$ . Here  $\omega_\xi$  is the vector state  $\langle \cdot, \xi, \xi \rangle$ . Let  $CPA(A, B(H))$  be the set of all triplets  $(\varrho, \eta, C)$  of a finite dimensional  $C^*$ -algebra  $C$  and unital completely positive maps  $A \xrightarrow[\eta]{\varrho} C \xrightarrow{\eta} B(H)$ . The von Neumann entropy of a state  $\psi$  on a finite dimensional  $C^*$ -algebra is denoted by  $S(\psi)$ . For a finite subset  $\omega \subset A$ , and a  $\delta > 0$ , put

$$scp_\phi(\pi; \omega, \delta) = \inf\{S(\omega_\xi \circ \eta) : (\varrho, \eta, C) \in CPA(A, B(H))$$

$$\text{and } \|\eta \circ \varrho(a) - \pi(a)\| < \delta\|a\|, \text{ for all } a \in \omega\}.$$

The  $scp_\phi(\pi; \omega, \delta)$  is defined to be  $\infty$  if no such approximation exists. Let

$$ht_\phi(\pi; \alpha, \omega, \delta) = \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} scp_\phi(\pi; \bigcup_{i=0}^{N-1} \alpha^i(\omega), \delta),$$

$$ht_\phi(\pi; \alpha, \omega) = \sup_{\delta > 0} ht_\phi(\pi; \alpha, \omega, \delta).$$

$$ht_\phi(\pi; \alpha) = \sup_{\omega} ht_\phi(\pi; \alpha, \omega).$$

**3.1.2. Remark.** In the case where a  $C^*$ -dynamical system  $(A, \alpha, \phi)$  has a faithful  $*$ -representation  $\pi : A \rightarrow B(H)$  and a cyclic unit vector  $\xi \in H$  such that  $\phi = \omega_\xi \circ \pi$ , we can prove that the value  $scp_\phi(\pi; \omega, \delta)$  does not depend of the choice of  $\pi$ .

**3.1.3. Remark.** A unital  $C^*$ -algebra  $A$  is exact if and only if for some  $C^*$ -algebra  $B$  there exists an embedding  $\iota : A \rightarrow B$  which is nuclear, that is, for arbitrary  $\epsilon > 0$  and for every finite set  $\omega \subset A$  there exist a finite dimensional  $C^*$ -algebra  $C$  and unital completely positive maps  $A \xrightarrow[\eta]{\rho} C \xrightarrow{\eta} B$  such that  $\|\iota(a) - \eta \circ \rho(a)\| < \epsilon\|a\|$

for all  $a \in \omega$ . ([Kir : Theorem 4.1], [Was]). Let  $(H_\phi, \pi_\phi, \xi_\phi)$  be the GNS(Gelfand-Naimark-Segal representation)-triplet associated with a state  $\phi$  of an exact  $C^*$ -algebra  $A$ . Consider the completely positive extension  $\varrho$  of the map  $\pi_\phi \circ \iota^{-1} : \iota(A) \rightarrow B(H_\phi)$  to  $B$  ([Ar]), then  $A \xrightarrow[\rho]{} C \xrightarrow[\varrho \circ \eta]{} B(H_\phi)$  implies the nuclearity of  $\pi_\phi$  (so that the approximation approach for  $scp_\phi(\omega; \delta)$  is reasonable).

**3.1.4. Definition.** Let  $(A, \alpha, \phi)$  be a  $C^*$ -dynamical system, where  $A$  is exact and the GNS-representation  $\pi_\phi$  is faithful. We define the entropy  $ht_\phi(\alpha)$  by  $ht_\phi(\pi_\phi, \alpha)$ .

The faithfulness of  $\pi_\phi$  is not always necessary in the definition of  $ht_\phi(\alpha)$ , but it is essential when we discuss the entropy of the free product  $\alpha * \beta$  of two automorphisms  $\alpha$  and  $\beta$ .

**3.1.5. Remark.** In the case where  $A$  is nuclear, this entropy coincides with that defined in in [Ch4], and in the form of  $scp_\phi(\omega; \delta)$ , we only need triplets  $(\varrho, \eta, C)$ , where  $C$  is a finite dimensional  $C^*$ -algebra, and  $A \xrightarrow[\varrho]{} C \xrightarrow[\eta]{} A$  are unital completely positive maps .

**3.1.6. Proposition.** *Let  $(A, \alpha, \phi)$  be a  $C^*$ -dynamical system such that  $A$  is exact and  $\phi$  has the faithful GNS-representation. Then the  $ht_\phi(\alpha)$  takes the value between the Connes-Narnhofer-Thirring entropy  $h_\phi(\alpha)$  and the Brown- Voiculescu's topological entropy  $ht(\alpha)$  :*

$$h_\phi(\alpha) \leq ht_\phi(\alpha) \leq ht(\alpha).$$

*If  $A$  is abelian, then we have*

$$h_\phi(\alpha) = ht_\phi(\alpha).$$

**3.1.7. Theorem.** For each  $i \in I$ , let  $A_i$  be a unital exact  $C^*$ -algebra, and let  $\phi_i$  be a state of  $A_i$  whose GNS-representation  $\pi_i$  is faithful. Let  $A$  and  $\phi$  be the  $C^*$ -algebra and the state given by the reduced free product construction :

$$(A, \phi) = \bigstar_{i \in I} (A_i, \phi_i).$$

If  $\alpha_i \in \text{Aut}(A_i)$  satisfies  $\phi_i \circ \alpha_i = \phi_i$  for all  $i \in I$ , then free product automorphism  $\alpha = \bigstar_{i \in I} \alpha_i \in \text{Aut}(A)$  preserves the state  $\phi$  and

$$ht_\phi(\alpha) = \sup_{i \in I} ht_{\phi_i}(\alpha_i).$$

**3.1.8. Remark.** About the topological entropy, Brown-Dykema-Shlyakhtenko proved in [BDS : Theorem 5.7] that

$$ht_\phi(\alpha) = \sup_{i \in I} ht_{\phi_i}(\alpha_i)$$

under the same conditions as in Theorem 3.1.7.

**3.2.** Now we discuss on the Connes-Størmer entropy  $H(\cdot)$  for each automorphism  $\alpha_T^s$  of  $N_s(I)$  ( $T \in GL(n, \mathbb{Z})$ ). Here  $\alpha^s$  is the action of  $GL(n, \mathbb{Z})$  on  $N_s(I)$  defined in 2.3.3, and  $N_s(I)$  is the hyperfinite  $\text{II}_1$  factor  $R$  if  $|I| = 1$  and is the free group factor  $L(F_m)$  if  $2 \leq |I| = m \leq \infty$ .

**3.2.1.** We denote by  $C_r^*(A(I), \mu_s)$  the  $C^*$ -algebra generated by  $\lambda^s(A(I))$  with respect to the  $\mu_s$ -representation  $\lambda^s$  of  $G(I)$  in 2.3.1, and by  $\tau$  the tracial state of  $C_r^*(A(I), \mu_s)$  given by  $\tau(\lambda^s(g)) = 0, (e \neq g \in G(I))$ . For a  $T \in GL(n, \mathbb{Z})$ , we denote by  $\beta_T^s$  the automorphism of  $C_r^*(A(I), \mu_s)$  induced from the automorphism  $\alpha_{\eta(T)}$  of  $A(I)$  defined in 2.1.2. The  $C^*$ -algebra  $C_r^*(A(I), \mu_s)$  is weakly dense in the factor  $N_s(I)$ , and the automorphism  $\alpha_T^s$  of  $N_s(I)$  is the extension of  $\beta_T^s$ .

**3.2.2. Proposition.** *Let  $n \geq 2$ , and let  $T \in GL(n, \mathbb{Z})$ . For all  $s \in [0, \pi/2]$  irrational mod.  $2\pi$ , the Connes-Størmer entropy  $H(\cdot)$ , Connes-Narnhofer-Thirring entropy  $h_\tau(\cdot)$ , Brown-Voiculescu's topological entropy  $ht(\cdot)$  and  $ht_\tau(\cdot)$  satisfy that*

$$H(\alpha_T^s) = h_\tau(\beta_T^s) \leq ht_\tau(\beta_T^s) \leq ht(\beta_T^s) \leq \log\left(\prod_{i=1}^n \mu_i \mu'_i\right).$$

Here  $\mu_i = \max(|\lambda_i|, 1)$  and  $\mu'_i = \max(\frac{1}{|\lambda_i|}, 1)$ , for the eigenvalue list  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  of the matrix  $T \in GL(n, \mathbb{Z})$ .

*In particular if  $T \in SL(n, \mathbb{Z})$ , then*

$$\log(\max\{\prod_{i=1}^n \mu_i, \prod_{i=1}^n \mu'_i\}) \leq H(\alpha_T^s) = h_\tau(\beta_T^s) \leq ht_\tau(\beta_T^s) \leq ht(\beta_T^s) \leq \log(\prod_{i=1}^n \mu_i \mu'_i).$$

**3.2.3. Remark.** We gave actions  $\{\alpha^s\}_s$  in Section 2, in order to obtain a non cocycle conjugate continuous family of actions on the free group factors. However, from a point of view of entropy theory, it would be interesting to treat the action such that we can get the exact value of the entropy for each automorphism appearing the action. As an example of such an action on the free group factors, we have the followings :

Let  $I \subset \mathbb{Z}$ , and let  $A_i$  be the copy of the the group  $C^*$ -algebra  $C^*(\mathbb{Z}^n)$ . Denote by  $\tau_i$  the tracial state of  $C_i$  taking 0 for  $g \in \mathbb{Z}^n, g \neq 1$ . Consider the reduced free product  $(C, \tau) = \ast_{i \in I} (C_i, \tau_i)$ . Let  $M_i$  be the von Neumann algebra generated by  $\pi_i(C_i)$ , where  $\pi_i$  is the GNS-representation by  $\tau_i$ , and let  $M$  be the von Neumann algebra generated by  $\pi_\tau(C)$ . Then  $M$  is isomorphic to  $L(F_m)$  by Dykema [D2 : Corollary 5.3], where  $m = |I|$ . Let  $\gamma_{i,T}$  be the automorphism of  $C_i$  induced by  $T \in SL(n, \mathbb{Z})$ , then  $\tau_i \cdot \gamma_{i,T} = \tau_i$ , and we have the automorphism  $\gamma_T = \ast_{i \in I} \gamma_{i,T}$  of  $C$  such that  $\tau \cdot \gamma = \tau$  (cf. [Ch3], [BD]). By the proof of Proposition 3.2.2,

$$h_\tau(\gamma_T) \leq ht_\tau(\gamma_T) = ht_{\tau_i}(\gamma_{i,T}) = h_{\tau_i}(\gamma_{i,T}) = \log\left(\prod_{i=1}^n \mu_i\right),$$

where  $\{\mu_i; 1 \leq i \leq n\}$  are the same as in Proposition 3.2.2. We denote by  $\hat{\gamma}_T$  (resp.  $\hat{\gamma}_{i,T}$ ) the extension of  $\gamma_T$  (resp.  $\gamma_{i,T}$ ) to  $M$  (resp.  $M_i$ ). Then

$$\log\left(\prod_{i=1}^n \mu_i\right) = H(\hat{\gamma}_{i,T}) \leq H(\hat{\gamma}_T) = h_\tau(\gamma_T).$$

Thus we have the action  $\hat{\gamma}$  of  $SL(n, \mathbb{Z})$  on  $L(F_m)$  such that

$$H(\hat{\gamma}_T) = \log\left(\prod_{i=1}^n \mu_i\right), \quad \forall T \in SL(n, \mathbb{Z}).$$

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